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1983 J. Phys. A: Math. Gen. 16 907

(http://iopscience.iop.org/0305-4470/16/5/010)

J. Phys. A: Math. Gen. 16 (1983) 907–912. Printed in Great Britain

Magnetic and gravitational moments of higher spin particles

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Received 6 April 1982, in final form 22 October 1982

Abstract. Consistent equations for spin-S particles in an electromagnetic field discovered recently by Buchdahl are shown in a non-relativistic approximation to describe particles with a gyromagnetic ratio equal to the reciprocal of the spin. The corresponding equations for a particle in a Riemann space are derived and the field-intrinsic spin interaction discussed. It is shown how the equations may be modified, without altering their consistency, to give the particles arbitrary gyromagnetic and 'gyrogravitational' ratios. Finally, in the gravitational case, it is shown that the equations have the most simple behaviour under conformal transformations when the ratio is the reciprocal of the spin. However, conformal invariance only obtains for spin-zero particles and for massless spin-$\frac{1}{2}$ and spin-1 particles, in which cases one has the usual equations.

1. Introduction

The metric has signature -2. The Infeld–van der Waerden symbols and the metric spinor satisfy

$$\sigma^{kA'B}\sigma^{l}{}_{A'B} = g^{kl} \tag{1.1}$$

$$\sigma^{kA'C}\sigma_k{}^{B'D} = \varepsilon^{A'B'}\varepsilon^{CD} \tag{1.2}$$

$$\varepsilon^{AB}\varepsilon_{BC} = -\delta^{A}{}_{C}. \tag{1.3}$$

This notation is used for typographical convenience. An important tensor–spinor is defined by

$$S^{klA}{}_{B} := \sigma^{[kC'A}\sigma^{l]}{}_{C'B}. \tag{1.4}$$

Formulae useful for the manipulation of these spinors and tensor–spinors are given by Buchdahl (1962). Note that symmetrising and antisymmetrising brackets act upon only one type of index, for example upon k and l in (1.4). (As it is not apparently in common use, I might remark that the use of the S tensor–spinor facilitates tensor–spinor transcriptions. For instance, the spinor curvature quantities can be directly written as

$$\begin{aligned}\Psi_{ABCD} &:= \tfrac{1}{4}S^{kl}{}_{AB}S^{mn}{}_{CD}R_{klmn}\\ &= \tfrac{1}{4}S^{kl}{}_{AB}S^{mn}{}_{CD}C_{klmn} + \tfrac{1}{12}R\varepsilon_{C(A}\varepsilon_{B)D}\end{aligned} \tag{1.5}$$

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$$E_{A'B'CD} := \tfrac{1}{4} S^{kl}{}_{A'B'} S^{mn}{}_{CD} R_{klmn}$$
$$= \tfrac{1}{2} \sigma^{kC'A} \sigma^{lD'B} E_{kl} \tag{1.6}$$

in a Riemann space.) The consistent equations of Buchdahl (1982) for a spin-S particle in a Riemann space, with ξ and η symmetric in their unprimed indices and $n := 2S$, are

$$\nabla^{B'}{}_{A_1} \xi^{A_1 A_2 \ldots A_n} = \kappa \eta^{B' A_2 \ldots A_n} \tag{1.7}$$

$$\nabla_{B'}{}^{A_1} \eta^{B' A_2 \ldots A_n} = \kappa \xi^{A_1 \ldots A_n} + \frac{n-1}{n\kappa} \varepsilon^{A_1 (A_2} S^{kl}{}_{BC} \xi^{A_3 \ldots A_n) BC}{}_{;kl}. \tag{1.8}$$

$\kappa := mc/\hbar\sqrt{2}$ and $\nabla^{A'B}$ is defined by

$$\nabla^{A'B} := \sigma^{kA'B} (\ )_{;k}. \tag{1.9}$$

Equation (1.8) gives a subsidiary condition by the symmetry of ξ in A_1 and A_2:

$$\nabla_{B'A_2} \eta^{B' A_2 \ldots A_n} = \frac{1}{\kappa} S^{kl}{}_{BC} \xi^{A_3 \ldots A_n BC}{}_{;kl}. \tag{1.10}$$

The consistency of (1.7) and (1.8) is shown by substituting for η in (1.10) from (1.7)—one obtains an identity. Replacing all covariant derivatives by electromagnetic derivatives $\partial_k - (\mathrm{i}e/\hbar c) A_k$ in (1.7) and (1.8) gives one consistent flat-space equations for a particle in an electromagnetic field.

Eliminating η from equations (1.7) and (1.8), using the decomposition of the Riemann tensor and the definition of the 'Weyl spinor'

$$C_{ABCD} := \tfrac{1}{4} S^{kl}{}_{AB} S^{mn}{}_{CD} C_{klmn} \tag{1.11}$$

one obtains (Buchdahl 1982, equation (6.2))

$$\left(\Box + \frac{m^2 c^2}{\hbar^2} + \frac{1}{12}(n+2) R\right) \xi^{A_1 \ldots A_n} = -2(n-1) C_{BC}{}^{(A_1 A_2} \xi^{A_3 \ldots A_n) BC} \tag{1.12}$$

where, in this case,

$$\Box := g^{kl} (\ )_{;kl}.$$

The corresponding equation in the electromagnetic case is

$$\left(\Box + \frac{m^2 c^2}{\hbar^2}\right) \xi^{A_1 \ldots A_n} = \frac{\mathrm{i}e}{\hbar c} F_{kl} S^{kl(A_1}{}_B \xi^{A_2 \ldots A_n) B} \tag{1.13}$$

where, in this case,

$$\Box := \eta^{kl} \left(\partial_k - \frac{\mathrm{i}e}{\hbar c} A_k\right)\left(\partial_l - \frac{\mathrm{i}e}{\hbar c} A_l\right).$$

(For $n = 0$ the terms on the right-hand sides of equations (1.12) and (1.13) are absent.) These two equations are, of course, not parity invariant.

2. The electromagnetic case

Treating ξ as the symmetrised product of n spin-$\frac{1}{2}$ 2-spinors one may define, using Clebsch–Gordan coefficients, an $(n+1)$-component vector Ψ from the components of ξ. (For example, for a spin of one define $(\Psi^1, \Psi^2, \Psi^3) := [\xi^{11}, (1/\sqrt{2})(\xi^{12}+\xi^{21}), \xi^{22}]$.) Now, one has the matrix relations

$$S^{ab} = \tfrac{1}{2}\mathrm{i}\varepsilon^{cab}\sigma_c \qquad S^{a4} = \tfrac{1}{2}\sigma^a \tag{2.1}$$

with $a, b, c = 1, 2, 3$ and the σ^a the usual Pauli matrices ($\sigma^1 = \left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$, etc). Then (1.13) can be written

$$\left(\frac{\hbar^2}{2m}\Box + \frac{1}{2}mc^2\right)\Psi = \frac{1}{S}\frac{e}{2mc}\left(\boldsymbol{B} + \frac{\mathrm{i}}{c}\mathbf{E}\right)\cdot \boldsymbol{S}\Psi \tag{2.2}$$

with $\boldsymbol{S} = (S_1, S_2, S_3)$ the spin-S angular momentum matrices. For 'weak' fields, and in a non-relativistic limit, put

$$\Psi = \exp[-(\mathrm{i}/\hbar)mc^2t]\Phi \tag{2.3}$$

where Φ varies slowly in time (i.e., consider states 'close' to the free particle, positive energy eigenstates—the 'single particle' states). The field is weak in the sense that it is assumed that the probability of particle–antiparticle 'fluctuations' due to the field is negligible, and hence it can be assumed that one is dealing with a so-called non-relativistic single particle. $E/c \ll B$ is also taken to be the case. Then, with $A_4 = 0$ for simplicity,

$$\frac{\partial^2}{\partial t^2}\Psi \simeq \left(-\frac{m^2c^4}{\hbar^2}\Phi - \frac{2\mathrm{i}mc^2}{\hbar}\frac{\partial\Phi}{\partial t}\right)\exp\left(-\frac{\mathrm{i}}{\hbar}mc^2t\right) \tag{2.4}$$

and so in this limit (2.2) becomes

$$-\frac{\hbar^2}{2m}\left(\boldsymbol{\nabla} - \frac{\mathrm{i}e}{\hbar c}\boldsymbol{A}\right)^2\Phi - \frac{1}{S}\frac{e}{2mc}\boldsymbol{B}\cdot\boldsymbol{S}\Phi = \mathrm{i}\hbar\frac{\partial\Phi}{\partial t} \tag{2.5}$$

which is the non-relativistic Schrödinger equation for a particle of charge $-e$ in a magnetic field, with gyromagnetic ratio $1/S$ (in Bohr magnetons). I remark that the values $1/S$ were conjectured by Belinfante (1953) and have been the subject of investigations since (e.g. Hagen and Hurley 1970).

3. The gravitational case

Consider a point in space–time, and an inertial reference frame at that point. Then it is well known that the conformal tensor in such a frame can be written

$$C_{klmn} = \begin{pmatrix} B & -\frac{1}{c}E \\ -\frac{1}{c}E & -B \end{pmatrix} \tag{3.1}$$

with B, E symmetric, traceless, real 3×3 matrices. (The indices 1–6 have replaced the pairs 23, 31, 12, 14, 24 and 34 respectively.) The coordinates have the dimension of time, whilst g_{kl} is dimensionless. C_{klmn} has the dimensions (time)$^{-2}$. Proceeding as in the electromagnetic case, one finds that (1.12) may be written

$$\left(\frac{\hbar^2}{2m}\Box + \frac{1}{2}mc^2 + \frac{n+2}{24m}\hbar^2R\right)\Psi = \frac{1}{S}\frac{1}{4m}\left(B_{ab} + \frac{\mathrm{i}}{c}E_{ab}\right)S^aS^b\Psi. \tag{3.2}$$

Using the approximation (2.3) one then obtains the appropriate non-relativistic Schrödinger equation (with reference to the local inertial frame)

$$\left(\frac{-\hbar^2}{2m}\nabla^2+\frac{n+2}{24m}\hbar^2 R\right)\Phi-\frac{1}{4mS}B_{ab}S^a S^b\Phi=\mathrm{i}\hbar\frac{\partial\Phi}{\partial t} \tag{3.3}$$

(assuming $E_{ab}/c \ll B_{ab}$). Define $\mu := (2mS)^{-1}$. Then one may consider the Hamiltonian

$$H=\frac{-\hbar^2}{2m}\nabla^2-\frac{1}{2}\mu B_{ab}S^a S^b+\frac{n+2}{24m}\hbar^2 R. \tag{3.4}$$

Then $[H, S^2]=0$, and

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle S^c\rangle = -\mu\varepsilon^{ca(d}B_a{}^{b)}\langle S_b S_d\rangle. \tag{3.5}$$

The simplest classical equation corresponding to this is

$$\dot{\boldsymbol{S}} = -\mu(\boldsymbol{BS})\times\boldsymbol{S}. \tag{3.6}$$

This equation is formally similar to that describing the torque-free motion of a rigid body about its centre of mass (Goldstein 1980)

$$\dot{\boldsymbol{L}} = -(I^{-1}\boldsymbol{L})\times\boldsymbol{L} \tag{3.7}$$

with $\boldsymbol{L}$ the angular momentum of the body, and I its moment of inertia tensor (assumed invertible). That μB and I^{-1} should be analogous is perhaps not surprising—I_{ab} measures the resistance of a body to being turned (loosely speaking) and m^{-1} times the field (together with a 'nonclassical' factor $(2S)^{-1}$) is then the 'inverse' of this. I remark that (3.6) has a simple solution for a type ID Weyl tensor, when the canonical form of B is diag$(\alpha, \alpha, -2\alpha)$ where α is a real number at each point in space–time. In this case (3.6) describes simple precession of the spin about the third spatial axis. The relationship between this discussion and the classical theory of a particle with spin in a gravitational field remains to be investigated.

4. Arbitrary ratios

Now observe that the consistency of (1.7) and (1.8) is unaltered by the addition of an arbitrary symmetric spinor of valence n to the right-hand side of (1.8). In particular, the addition of the term

$$\kappa^{-1}(1-\lambda S)S^{kl(A_1}{}_B\xi^{A_2\ldots A_n)B}{}_{;kl}, \tag{4.1}$$

where λ may depend on S, leads to the following equation instead of (1.12) $(n>0)$:

$$\left(\Box+\frac{m^2c^2}{\hbar^2}+\frac{1}{24}n(n+2)\lambda R\right)\xi^{A_1\ldots A_n} = -n(n-1)\lambda C_{BC}{}^{(A_1A_2}\xi^{A_3\ldots A_n)BC}. \tag{4.2}$$

Similarly, in the electromagnetic case, one obtains instead of (1.13) the equation

$$\left(\Box+\frac{m^2c^2}{\hbar^2}\right)\xi^{A_1\ldots A_n} = \frac{n\lambda}{2}\frac{\mathrm{i}e}{\hbar c}F_{kl}S^{kl(A_1}{}_B\xi^{A_2\ldots A_n)B}. \tag{4.3}$$

Proceeding as before one finds λ to be the gyromagnetic ratio of the particle. One may then call the λ in (4.2) the 'gyrogravitational ratio'.

5. Conformal invariance

One finds that the transformation of equation (4.2) under a conformal transformation of the metric is most simple when $\lambda = 1/S$. Unfortunately, however, even then the equation is not conformally invariant in general. By 'conformally invariant' I mean that the equation goes into another of the same form under the transformation

$$\begin{aligned} g^{kl} &\to \hat{g}^{kl} = \exp(-2q) g^{kl} \\ \sigma^{kA'B} &\to \hat{\sigma}^{kA'B} = \sigma^{kA'B} \\ \varepsilon_{AB} &\to \hat{\varepsilon}_{AB} = \exp(-q)\varepsilon_{AB} \\ \xi &\to \hat{\xi} = \exp(-\nu q)\xi \end{aligned} \tag{5.1}$$

where q is an arbitrary (real) function and ν a real number (the 'conformal weight' of ξ). The transformed spin connection is given by (Buchdahl 1959)

$$\hat{\Gamma}^{A}{}_{Bk} = \Gamma^{A}{}_{Bk} - \sigma_{kC'B}\sigma_l{}^{C'A} q^{;l}. \tag{5.2}$$

The choice $\nu = -1$ is convenient. The mass is given the conformal weight -1. One finds that

$$\hat{R} = \exp(-3q)[R + 6(q_{;k}{}^{k} + q_{;k}q^{;k})]. \tag{5.3}$$

After some calculation, one then finds that the conformal transform of (4.2) is (in natural units)

$$\begin{aligned} &[\Box + \tfrac{1}{24}n(n+2)\lambda R + m^2 + \tfrac{1}{2}(n+2)(\tfrac{1}{2}n\lambda - 1)(q_{;k}{}^{k} + q_{;k}q^{;k})]\xi^{A_1\ldots A_n} \\ &\qquad - 2n[\nabla^{B'(A_1}q]\nabla_{B'C}\xi^{A_2\ldots A_n)C} \\ &\quad = -n(n-1)\lambda C_{BC}{}^{(A_1A_2}\xi^{A_3\ldots A_n)BC}. \end{aligned} \tag{5.4}$$

Note that C^{ABCD} is a conformal invariant. Thus choose $\lambda = 2/n = 1/S$, giving

$$\begin{aligned} &[\Box + m^2 + \tfrac{1}{12}(n+2)R]\xi^{A_1\ldots A_n} \\ &\quad = -2(n-1)C_{BC}{}^{(A_1A_2}\xi^{A_3\ldots A_n)BC} + 2n[\nabla^{B'(A_1}q]\nabla_{B'C}\xi^{A_2\ldots A_n)C}. \end{aligned} \tag{5.5}$$

The function q is arbitrary, and thus (4.2) is conformally invariant only if $\lambda = 1/S$ and

$$\nabla_{B'C}\xi^{A_2\ldots A_nC} = 0. \tag{5.6}$$

Thus the mass must be taken to be zero. (Equation (5.6) then implies $\lambda = 1/S$ in (4.2) with $m = 0$.) However, as is well known, for spin $\geqslant \frac{3}{2}$ equation (5.6) leads to the subsidiary condition

$$S^{kl}{}_{BC}\xi^{A_3\ldots A_nBC}{}_{;kl} = 0 \tag{5.7}$$

which must be satisfied by the field, and the equation is then unsuitable for describing the particle. Thus, equation (4.2) is conformally invariant only for $m = 0$, $\lambda = 1/S$ and only for spins $\frac{1}{2}$ and 1. In these two cases one has the second-order wave equations

$$\begin{aligned} (\Box + \tfrac{1}{4}R)\Phi^{A} &= 0 \\ (\Box + \tfrac{1}{3}R)\Phi^{AB} &= -2C_{CD}{}^{AB}\Phi^{CD}. \end{aligned} \tag{5.8}$$

For spin zero one has, of course,

$$(\Box + m^2 + \tfrac{1}{6}R)\Phi = 0 \tag{5.9}$$

and the corresponding equation for a massless particle (equation (5.9) with $m = 0$).

Acknowledgments

It is a pleasure to thank Hans Buchdahl for many useful discussions. I would also like to thank one of the referees for helpful remarks. This work was supported by the Australian people, through a grant from the Government of Australia.

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